Efficient Time-Domain Electromagnetic Analysis Using CDF Biorthogonal Wavelet Expansion

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***Abstract* — The multi-resolution time-domain (MRTD) algorithm is implemented using Cohen-Daubechies-Feauveau (CDF) wavelet bases, resulting in a computationally efficient numerical scheme for electromagnetic field analysis. The application to a simple scattering problem demonstrates the advantages of the method versus the conventional finite-difference time-domain (FDTD) technique.**

I. Introduction

The multi-resolution time-domain (MRTD) method has been applied recently to a number of electromagnetic field problems, such as microwave cavities and structures [1-4], as well as scattering by various targets [5]. It was demonstrated that the method produces significant savings in computational resources *vis-à-vis* the conventional finite-difference time-domain (FDTD) scheme, for a given accuracy in the solution.

The essence of the MRTD algorithm is constituted by an expansion of the fields in a wavelet basis, followed by a Galerkin-type discretization of Maxwell's equations. Previous work on this method has concentrated on Haar [5], Battle-Lemarie [1,6] or orthonormal Daubechies wavelet families [2,3]. More recently, we introduced an expansion in terms of the Cohen-Daubechies-Feauveau (CDF) biorthogonal wavelets [4], in the quest for a balance between good numerical dispersion properties and algorithmic simplicity.

The MRTD technique allows the treatment of large electromagnetic problems with reduced computer resources by addressing two basic issues present in the context of FDTD: the numerical dispersion is decreased by achieving higher-order approximation of the derivatives, and the multi-resolution approach allows for denser discretization only in selected regions of the computational domain, while keeping low sampling rates in the regions of slow field variation. It should be mentioned that both these issues could be treated separately within the FDTD family of techniques [7]. However, significant problems are associated with media boundaries (for higher-order schemes) and instability (for multi-grid schemes), whenever alternatives to the classic Yee algorithm are implemented. In addition, the MRTD algorithm incorporates a sub-cell model for treating a non-conformal boundary between dielectric media, resulting in increased accuracy as compared to a staircase model of the same boundary [5].

In this paper we formulate the MRTD algorithm using CDF biorthogonal wavelets (Sec. II). In Sec. III we analyze different aspects of the algorithm and compare its performance with other existing MRTD schemes. In Sec. IV we apply the algorithm to a simple two-dimensional scattering problem and compare both the results and the computational resources involved with the traditional FDTD scheme. We summarize in Sec. V with conclusions and ideas for future applications.

II. Formulation Of The CDF-MRTD Scheme

A. Homogeneous medium

In order to keep the presentation simple, we derive the CDF biorthogonal MRTD scheme for one-dimensional propagation through a homogeneous medium. In the following, we consider the expansion of the field component E_z in terms of scaling functions and first level wavelet functions, in the spatial dimension, and rectangular pulse expansion (similar to Yee discretization) in the time dimension.

$$E_z(x,t) = \sum_{k,m=-\infty}^{\infty} \left[E^{\Phi}_{k,m} \tilde{\Phi}_m(x) + E^{\Psi}_{k,m} \tilde{\Psi}_m(x) \right] h_k(t) \quad (1)$$

Here, we denote by $\tilde{\Phi}_m$ the dual scaling function shifted by m units, and by $\tilde{\Psi}_m$ the first level dual wavelet function displaced by m units. For time discretization we use rectangular pulses $h_k(t)$, where k represents the shift in time units. A similar equation holds for H_y, only the supports of the scaling/wavelet functions are displaced half a unit relative to E_z. Throughout this paper, we use CDF wavelet families [8] for which the wavelet functions (and their duals) are symmetric about ½ (*e.g.*, CDF (2,2), CDF (2,4) or CDF (2,6)). In this case, the positions of the E^{Φ} and E^{Ψ} components are staggered at half of a cell interval.

The Galerkin discretization procedure is effected by testing Maxwell's equations with the scaling functions,

Φ_m, and the wavelet functions, Ψ_m, respectively [1]. The resulting update equations for the electric field scaling and wavelet expansion coefficients are:

$$E^{\Phi}_{k+1,m} = E^{\Phi}_{k,m} + \frac{\Delta t}{\varepsilon \Delta x}\left(\begin{array}{l} \sum_{i=1}^{n_a} a(i)\left(H^{\Phi}_{k,m+i-1} - H^{\Phi}_{k,m-i}\right) \\ + \sum_{i=1}^{n_c} c(i)\left(H^{\Psi}_{k,m+i} - H^{\Psi}_{k,m-i}\right) \end{array} \right) \quad (2.a)$$

$$E^{\Psi}_{k+1,m} = E^{\Psi}_{k,m} + \frac{\Delta t}{\varepsilon \Delta x}\left(\begin{array}{l} \sum_{i=1}^{n_d} d(i)\left(H^{\Phi}_{k,m+i} - H^{\Phi}_{k,m-i}\right) \\ + \sum_{i=1}^{n_b} b(i)\left(H^{\Psi}_{k,m+i} - H^{\Psi}_{k,m-i+1}\right) \end{array} \right) \quad (2.b)$$

where

$$a(i) = \int \frac{\partial \tilde{\Phi}_{m+i}(x)}{\partial x} \Phi_{m+1/2}(x) dx \quad (3.a)$$

$$b(i) = \int \frac{\partial \tilde{\Psi}_{m+i}(x)}{\partial x} \Psi_{m+1/2}(x) dx \quad (3.b)$$

$$c(i) = \int \frac{\partial \tilde{\Psi}_{m+i}(x)}{\partial x} \Phi_{m+1/2}(x) dx \quad (3.c)$$

$$d(i) = \int \frac{\partial \tilde{\Phi}_{m+i}(x)}{\partial x} \Psi_{m-1/2}(x) dx \quad (3.d)$$

The numbers n_a, n_b, n_c and n_d, are called stencil sizes and indicate the number of the non-zero coefficients in the MRTD scheme. A set of similar equations holds for the H_y scaling/wavelet coefficients. The extension to two or three dimensions is straightforward.

B. Inhomogeneous medium

Since the support of the scaling/wavelet function extends over a few cells, it is apparent that coupling should occur between the update equations for adjacent field coefficients, at the interface between two media. This results in matrix equations that need to be solved in order to simultaneously update the field coefficients next to the interface. Obviously, this approach significantly complicates the implementation of the MRTD algorithm.

However, we can simplify the formulation by recognizing the fact that we sample the field values at discrete spatial location, and using the interpolating property of the dual scaling function, *i.e.*:

$$\tilde{\Phi}_m(m') = \delta_{m-m'} \quad (4)$$

This property has been utilized in [2,3] in the context of an MRTD formulation based on orthonormal Daubechies wavelets. It has been shown that the Daubechies scaling functions approximately satisfy the shifted interpolation property, which means that we can consider point sampling of the field at integer locations, with negligible error. However, for the CDF biorthogonal dual scaling function with support 2 (like in CDF (2,2), CDF (2,4), etc.), the interpolation property holds *exactly*, without the need of shifting the basis functions. This point-wise sampling of the field allows a formulation similar to the Yee algorithm, in which the material properties are sampled at the current point.

III. Analysis Of The CDF-MRTD Scheme

A. Choice of the wavelet family

The main objective of the MRTD method is a minimization of the computational resources required for a given accuracy of the electromagnetic solution. In this context, we would like to reduce the number of unknowns, by decreasing the number of discretization points per wavelength, while simultaneously keeping the numerical dispersion under control. Regularity (smoothness) and vanishing moments of the wavelet functions are the main requirements in this case, which relates to a typical data compression problem [8]. A second and distinct issue involves reduction of the total number of computations required by the algorithm. For this purpose, the scaling/wavelet functions of choice should have minimum support. Also, in order to allow for a large time step at the stability limit, a large Courant number is desired.

Among wavelet bases previously considered in the literature, the Haar family yields a simple algorithm [5], which bears close similarities to the Yee FDTD scheme. Unfortunately, the Haar wavelets lack smoothness.

The Battle-Lemarie family of wavelets, which are derived from *B*-spline functions [8], have good regularity properties (depending on the order of the spline functions used in design), but they have infinite support. This results, theoretically, in an infinite number of MRTD terms in each update equation. Since the Battle-Lemarie functions display exponential decay, the higher-order MRTD coefficients also decay fast. Nevertheless, truncating the sequence of MRTD coefficients [1] to a reasonable number (usually 8-12 on each side) poses problems in terms of arithmetic precision, by vitiating the properties of the wavelet functions imposed by design. Also, the relatively large stencil size and the small Courant number make the algorithm inefficient in terms of computational complexity.

Another possible choice, which was already mentioned, is that of Daubechies orthogonal wavelets [2,3]. It is interesting to point out that, at the level of a scaling function field expansion, the MRTD schemes based on Daubechies wavelets of order 4, 6, 8 etc. are equivalent with the schemes based on CDF biorthogonal wavelets of order (2,2), (2,4), (2,6) etc. (compare the coefficients listed in [3] and [4]).

TABLE I
COURANT NUMBER AT THE STABILITY LIMIT FOR THE MRTD ALGORITHM IN ONE DIMENSION

	CDF (2,2)	CDF (2,4)	CDF (2,6)	Cubic spline Battle-Lemarie
Scaling only	0.7500	0.6844	0.6585	0.6371
Scaling + One Level Wavelet	0.6046	0.4831	0.4221	0.2625

B. Stability and dispersion analysis

The stability criterion for the general MRTD algorithm can be found in [6]. In Table I we list the values of the Courant number at the stability limit for various MRTD schemes, considering one-dimensional propagation. In two or three dimensions, these values must be adjusted by factors of $1/\sqrt{2}$ and $1/\sqrt{3}$, respectively [7]. Notice that the low-order CDF (2,2) scheme has a Courant number higher than the other schemes, and the difference becomes significant when we consider one level of wavelets in the expansion.

A detailed account of the numerical dispersion analysis for the CDF-MRTD scheme was given in [4]. It should be mentioned that the dispersion performance depends on several parameters, including the spatial sampling rate, the Courant number, the angle of propagation and the number of wavelet levels considered in the expansion. In general, MRTD schemes can operate at sampling rates at least two times less than FDTD, when the same dispersion error is tolerated. Also, it can be shown that the dispersion error for all MRTD schemes decreases together with the Courant number (the further we are from the stability limit, the better the performance). This suggests that, for the same Courant number, when both scaling and first level wavelet functions are included in the expansion, the low-order CDF (2,2) family, which has a large stability limit, delivers better dispersion performance than the other wavelet bases [4].

C. Computational complexity

The number of floating-point operations for the update equations is related to the stencil sizes, n_a, n_b, n_c and n_d. These numbers are listed in Table II for several wavelet bases. It is clear from this table that the low-order CDF schemes perform one field update step more efficiently than the Battle-Lemarie scheme. Also, the larger Courant number means that we can run a stable code with larger time step, thereby reducing the total number of time steps required in a simulation.

TABLE II
STENCIL SIZES FOR DIFFERENT MRTD SCHEMES

	CDF (2,2)	CDF (2,4)	CDF (2,6)	Cubic spline Battle-Lemarie coefficients truncated at 10^{-3}
n_a	3	5	7	9
n_b	3	5	7	9
n_c	2	3	4	8
n_d	3	6	9	8

IV. NUMERICAL RESULTS

In this section we apply the CDF (2,2) MRTD scheme to a simple two-dimensional scattering problem. The relevant physical and dimensional parameters are shown in Fig. 1. The excitation consists of a pulsed plane wave, with the incident waveform given by the 4^{th} order Rayleigh pulse, centered at 3 GHz. We consider TE (horizontal) polarization. The incidence angle is 45^0 and the observation is made in the backscatter direction, in the far zone. We compare the solutions obtained via the classic Yee FDTD algorithm with the MRTD solution. For FDTD, we use a discretization rate of 80 samples per central wavelength (λ_c) in air and a Courant number of 0.6. For the CDF-MRTD, we use a grid with 20 samples per central wavelength in air. We perform an expansion of the fields in terms of scaling functions throughout the entire computational domain, and use wavelets only in the inhomogeneity areas (thus doubling the resolution in these regions). The Courant number is taken 0.3, therefore the time step is twice as large as for FDTD. The total grid size for the FDTD is 440 x 200 cells, whereas for the MRTD is 110 x 50 cells. The number of time steps for FDTD is 4096, whereas for the MRTD is 2048. The wavelets cover about 12% of the MRTD computational domain, therefore, the total number of scaling and wavelet coefficients is about 1.36 times the total number of MRTD cells. In both implementations, we use PML absorbing boundary conditions [7].

The resulting far-zone time-domain waveforms are shown in Fig. 2. The match between the two methods is very good. When we compare the computer resources

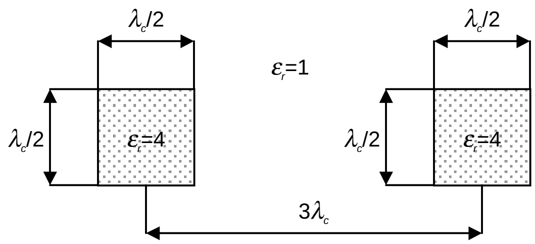

Fig. 1. The physical configuration of the scattering problem. It consists of two rectangular dielectric infinite cylinders, with ε_r=4, placed in free-space.

required by both implementations, we expect MRTD to utilize about 12 times less memory than FDTD, and to run about 6 times faster. However, our numerical experiments show that the increase in computational speed is more significant (typically, about 11 times). We attribute this to the fact that the MRTD update equations are more efficiently processed on the particular type of machine (Pentium III) that we used in our simulations.

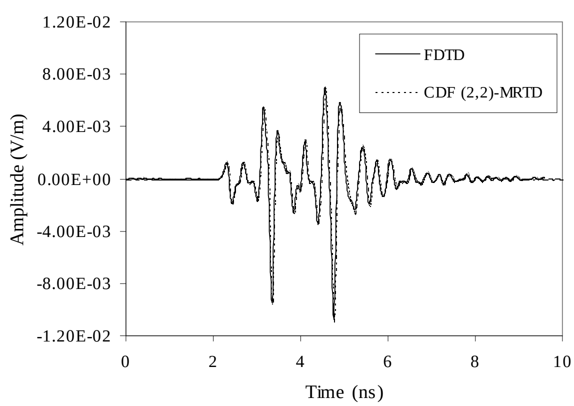

Fig. 2. Time-domain scattered field for the configuration in Fig. 1. The waveforms obtained with FDTD and CDF (2,2)-MRTD are one on top of the other.

V. Conclusions

In this paper we demonstrated the implementation of an electromagnetic scattering problem utilizing the CDF-MRTD scheme. We compared the results with those obtained by the traditional Yee FDTD algorithm, emphasizing the reduction in computational resources in the former case. There is no increase in algorithmic complexity for the CDF-MRTD method when applied to inhomogeneous media, due to the interpolating property of the dual scaling function. We envision the application of this type of schemes to more complicated configurations, especially for large electromagnetic problems. Moreover, it is easy to notice that the resource savings of MRTD as compared to FDTD are even greater in three dimensions. One particular application that may be of interest in the context of CDF-MRTD implementation is scattering by photonic band-gap structures.

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